Complex Analysis: Midterm Exam

Aletta Jacobshal 01, Monday 17 December 2018, 09:00–11:00
Exam duration: 2 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of the first page of each of your exam sheets and on the envelope. **Do NOT seal the envelope!**
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you **must** explain which results you are using and why the conditions for using such results are satisfied.
- 10 points are "free". There are 5 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.

Question 1 (15 points)

Prove that if a function $f(z) = u(x, y) + \mathrm{i}v(x, y)$ is differentiable at $z_0 = x_0 + \mathrm{i}y_0$ then the derivative is given by

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + \mathrm{i}\frac{\partial v}{\partial x}(x_0, y_0).$$

Solution

Since the derivative $f'(z_0)$ exists, by definition, the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

exists.

Let $z = x + \mathrm{i}y_0$ so that we also have $z - z_0 = x - x_0$. Then $z \to z_0$ implies $x \to x_0$ and thus we have that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{x \to x_0} \frac{u(x, y_0) + \mathrm{i}v(x, y_0) - u(x_0, y_0) - \mathrm{i}v(x_0, y_0)}{x - x_0}.$$

Re-arranging terms we find

$$f'(z_0) = \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + \mathrm{i} \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}.$$

By definition, the limits in the last expression are the corresponding partial derivatives of u and v with respect to x at the point (x_0, y_0), that is,

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + \mathrm{i}\frac{\partial v}{\partial x}(x_0, y_0).$$

Question 2 (20 points)

Prove that the function $f(z) = \sqrt{z+1}\sqrt{z-1}$ is discontinuous at $z \in (-1, 1)$, that is, along the open interval on the real axis between -1 and 1. You **must** check the limits of $f(z)$ at $z \in (-1, 1)$.

Here $\sqrt{z}$ is the principal value $e^{\frac{1}{2}\operatorname{Log} z}$ of the multi-valued $z^{1/2}$. It is given that for $-r < 0$ we have $\lim_{t \to 0^+} \sqrt{-r + \mathrm{i}t} = \mathrm{i}\sqrt{r}$ and $\lim_{t \to 0^-} \sqrt{-r + \mathrm{i}t} = -\mathrm{i}\sqrt{r}$.

Solution

We compute the limits $\lim_{y \to 0^+} f(x + \mathrm{i}y)$ and $\lim_{y \to 0^-} f(x + \mathrm{i}y)$ for $x \in (-1, 1)$.

First, for $x \in (-1, 1)$ we have

$$\lim_{y \to 0} \sqrt{z + 1} = \lim_{y \to 0} \sqrt{x + 1 + \mathrm{i}y} = \sqrt{x + 1},$$

since $x > -1$ implies $x + 1 > 0$ and Log (hence, also $\sqrt{}$) is continuous at $(0, \infty)$.

Then, for $x \in (-1, 1)$ we have

$$\lim_{y \to 0^+} \sqrt{z - 1} = \lim_{y \to 0^+} \sqrt{x - 1 + \mathrm{i}y} = \mathrm{i}\sqrt{1 - x},$$

where we used that y approaches 0 from above and since $x < 1$ we have $x - 1 < 0$. Similarly,

$$\lim_{y \to 0^-} \sqrt{z - 1} = \lim_{y \to 0^-} \sqrt{x - 1 + \mathrm{i}y} = -\mathrm{i}\sqrt{1 - x},$$

where here we used that y approaches 0 from below.

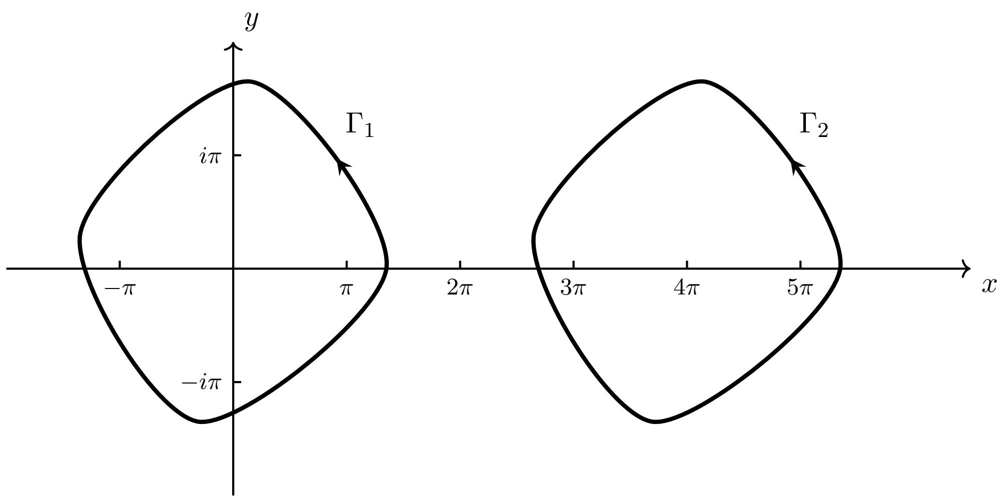
Therefore, for $x \in (-1, 1)$ we have

$$\lim_{y \to 0^+} f(x + \mathrm{i}y) = \mathrm{i}\sqrt{x + 1}\sqrt{1 - x} \neq \lim_{y \to 0^-} f(x + \mathrm{i}y) = -\mathrm{i}\sqrt{x + 1}\sqrt{1 - x},$$

and thus the function is discontinuous at $z \in (-1, 1)$.

Question 3 (20 points)

Compute $\displaystyle\int_\Gamma \frac{ze^z}{(z - \mathrm{i}\pi)^2}\,\mathrm{d}z$ first for the contour $\Gamma = \Gamma_1$ and then for the contour $\Gamma = \Gamma_2$ shown below.



Solution

To compute the integral along $\Gamma = \Gamma_1$ we use the generalized Cauchy integral formula with $n = 1$, $z_0 = \pi\mathrm{i}$ (which is inside Γ_1), and $f(z) = ze^z$ (which is an entire function and thus analytic on and inside Γ_1).

Then we find

$$\int_{\Gamma_1} \frac{f(z)}{(z-\pi\mathrm{i})^2}\,\mathrm{d}z = 2\pi\mathrm{i} f'(\pi\mathrm{i}).$$

The derivative is

$$f'(z) = (ze^z)' = (z+1)e^z,$$

and thus

$$f'(\pi\mathrm{i}) = (1+\pi\mathrm{i})e^{\pi\mathrm{i}} = -(1+\pi\mathrm{i}).$$

We conclude that

$$\int_{\Gamma_1} \frac{f(z)}{(z-\pi\mathrm{i})^2}\,\mathrm{d}z = -2\pi\mathrm{i}(1+\pi\mathrm{i}) = 2\pi^2 - 2\pi\mathrm{i}.$$

The integrand $ze^z/(z-\pi\mathrm{i})^2$ is analytic on the complex plane except the point $\pi\mathrm{i}$. In particular, it is analytic on and inside Γ_2. This implies that

$$\int_{\Gamma_2} \frac{ze^z}{(z-\pi\mathrm{i})^2}\,\mathrm{d}z = 0.$$

Question 4 (20 points)

Prove that on the positively oriented circle C given by $|z+1| = 2$ we have

$$\left|\int_C \frac{e^z}{\bar{z}-3}\,\mathrm{d}z\right| \leq 2\pi e.$$

NB: The inequalities that you use will either need to be proved (algebraically or geometrically) or to be part of the theory presented in the book. In particular, the triangle inequality, in its different forms, is considered known and so are inequalities between $\operatorname{Re} z$, $\operatorname{Im} z$ and $|z|$. Fewer points will be given if the necessary inequalities are established only “visually” (that is, by looking at the right picture but without complete proof).

Solution

We use the Estimation Lemma to find

$$\left|\int_C \frac{e^z}{\bar{z}-3}\,\mathrm{d}z\right| \leq M\ell(C),$$

where $\ell(C) = 4\pi$ is the length of the circle (which has radius 2), and M is any number such that

$$\left|\frac{e^z}{\bar{z}-3}\right| \leq M,$$

for all $z \in C$.

We need to find a value for M. For the numerator, e^z, we have

$$|e^z| = |e^x||e^{\mathrm{i}y}| = |e^x| = e^x.$$

On the circle C we have

$$x + 1 = \operatorname{Re}(z+1) \leq |z+1| = 2,$$

implying $x \leq 1$ and thus

$$|e^z| = e^x \leq e.$$

For the denominator we find

$$|\bar{z} - 3| = |z - 3| = |z + 1 - 4| \geq ||z + 1| - | - 4|| = |2 - 4| = 2.$$

Therefore,

$$\frac{1}{|\bar{z} - 3|} \leq \frac{1}{2}.$$

We conclude that for $z \in C$ we have

$$\left|\frac{e^z}{\bar{z} - 3}\right| \leq \frac{e}{2},$$

and thus from the Estimation Lemma with $M = e/2$ and $\ell(C) = 4\pi$ we find

$$\left|\int_C \frac{e^z}{\bar{z} - 3}\, \mathrm{d}z\right| \leq 2\pi e.$$

Question 5 (15 points)

A function $f(z)$ on $\mathbb{C}$ is **doubly periodic** if there are non-zero complex numbers ω_1 and ω_2 such that:

(a) $f(z + \omega_1) = f(z + \omega_2) = f(z)$ for all $z \in \mathbb{C}$, and

(b) ω_1 and ω_2 are linearly independent over the reals, that is, $\omega_2/\omega_1 \notin \mathbb{R}$, implying that each $z \in \mathbb{C}$ can be written as $z = \lambda_1\omega_1 + \lambda_2\omega_2$ with unique $\lambda_1, \lambda_2 \in \mathbb{R}$.

Prove that if a doubly periodic function is entire then it must be constant.

Hint: Suppose that $g : V \to \mathbb{R}$, where V is a closed and bounded subset of $\mathbb{C}$, is continuous. It is then known that there is $M > 0$ such that $-M \leq g(z) \leq M$ for all $z \in V$.

Solution

We consider the function f in the parallelogram L defined by the vertices 0, ω_1, ω_2, $\omega_1 + \omega_2$, including the vertices and the edges that connect them. Formally,

$$L = \{t_1\omega_1 + t_2\omega_2 \in \mathbb{C} \,:\, (t_1, t_2) \in [0, 1]^2\}.$$

L, being a parallelogram, is bounded. Moreover, because it includes its boundary points (the edges and vertices), it is closed.

Since $f(z)$ is entire, it is continuous on $\mathbb{C}$, and thus $|f(z)|$ is also continuous on $\mathbb{C}$, and also on the closed and bounded L. Therefore, there is M such that the continuous real-valued function $|f(z)|$ is bounded by M on L. In particular, $0 \leq |f(z)| \leq M$ for all $z \in L$.

For any other point $z \in \mathbb{C}$ write

$$z = \lambda_1\omega_1 + \lambda_2\omega_2,$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$. Then write $\lambda_1 = k_1 + t_1$ and $\lambda_2 = k_2 + t_2$ where k_1, k_2 are integers and $t_1, t_2 \in [0, 1)$. This implies

$$z = (t_1\omega_1 + t_2\omega_2) + (k_1\omega_1 + k_2\omega_2).$$

Since $t_1, t_2 \in [0,1)$ we conclude that $z_0 = t_1\omega_1 + t_2\omega_2 \in L$. We have shown that for any point $z \in \mathbb{C}$ there is a point $z_0 \in L$ and integers k_1, k_2 such that

$$z = z_0 + k_1\omega_1 + k_2\omega_2.$$

Therefore,

$$f(z) = f(z_0 + k_1\omega_1 + k_2\omega_2) = f(z_0),$$

and thus

$$0 \le |f(z)| = |f(z_0)| \le M,$$

since $z_0 \in L$.

This means that $f(z)$ is a bounded entire function and Liouville's theorem implies that it must be constant.

Formulas

The Cauchy-Riemann equations for a function $f = u + \mathrm{i}v$ are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The principal value of the logarithm is

$$\operatorname{Log} z = \operatorname{Log}|z| + \mathrm{i}\operatorname{Arg} z.$$

The generalized Cauchy integral formula is

$$f^{(n)}(z_0) = \frac{n!}{2\pi\mathrm{i}} \int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$